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On the norm theorem for semisingular quadratic forms ☆

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ABSTRACT

The aim of this paper is to prove some results concerning the norm theorem for semisingular quadratic forms, i.e., those which are neither nonsingular nor totally singular. More precisely, we will give necessary conditions in order that an irreducible polynomial, possibly in more than one variable, is a norm of a semisingular quadratic form, and we prove that our conditions are sufficient if the polynomial is given by a quadratic form which represents 1. As a consequence, we extend the Cassels–Pfister subform theorem to the case of semisingular quadratic forms.

1. INTRODUCTION

Let F be a commutative field. A classical result in the algebraic theory of quadratic forms is the norm theorem which was proved by Knebusch in characteristic different from 2 [4]. This theorem asserts that, for $p \in F[x_1, \ldots, x_n]$ irreducible and normed, an anisotropic quadratic form φ becomes hyperbolic over $F(p)$, the quotient field of $F[x_1, \ldots, x_n]/(p)$, if and only if p is a norm of φ over $F(x_1, \ldots, x_n)$, i.e., φ is isometric to $p\varphi$ over $F(x_1, \ldots, x_n)$ (normed means that the coefficient of the highest monomial occurring in p with respect to the lexicographical ordering is 1).

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For the norm theorem in characteristic 2, we have to treat separately various types of quadratic forms. Before going further, we recall that any quadratic form φ of dimension $\geqslant 1$ can be written up to isometry:

$$(1) \qquad \varphi \cong [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1 \rangle \perp \cdots \perp \langle c_s \rangle,$$

where $\cong$ and $\perp$ denote the isometry and orthogonal sum of quadratic forms, and $[a, b]$ (resp. $\langle a \rangle$) denotes the quadratic form $ax^2 + xy + by^2$ (resp. ax^2). As in (1), the quadratic form $\langle c_1 \rangle \perp \cdots \perp \langle c_s \rangle$ is unique up to isometry since it is the restriction of φ to its radical. We call it the quasilinear part of φ and denote it by $\mathrm{ql}(\varphi)$. We say that the quadratic form φ is:

- nonsingular if $\dim \mathrm{ql}(\varphi) = 0$,
- totally singular if $\dim \varphi = \dim \mathrm{ql}(\varphi)$,
- semisingular if $\dim \varphi > \dim \mathrm{ql}(\varphi) > 0$.

Clearly, these three types of quadratic forms are pairwise exclusive.

A quadratic form φ is called singular if $\dim \mathrm{ql}(\varphi) > 0$.

Let us also recall the Witt decomposition [3, Proposition 2.4], which means that any nonzero quadratic form φ is uniquely decomposed as follows:

$$\varphi \cong \varphi_{\mathrm{an}} \perp i \times \mathbb{H} \perp j \times \langle 0 \rangle,$$

where $\mathbb{H} = [0, 0]$ is the hyperbolic plane, φ_{an} is an anisotropic quadratic form that we call the anisotropic part of φ (here $n \times \psi$ denotes the sum of n copies of ψ). The integer i (resp. j) is called the Witt index of φ and denoted by $i_W(\varphi)$ (resp. the defect index of φ and denoted by $i_d(\varphi)$). The total index of φ, denoted by $i_t(\varphi)$, is the integer $i_W(\varphi) + i_d(\varphi)$.

A nonsingular (resp. singular) quadratic form φ is called hyperbolic (resp. quasi-hyperbolic) if $i_W(\varphi) = \frac{\dim \varphi}{2}$ (resp. $\dim \varphi$ is even and $i_t(\varphi) \geqslant \frac{\dim \varphi}{2}$). This definition of quasi-hyperbolicity is invariant under field extensions. In the case of totally singular forms, we get a slightly different definition from the original one fixed in [6], which says that $i_d(\varphi) = \frac{\dim \varphi}{2}$. Moreover, as was noted in [8, beginning of Section 3], all the results proved in [6] remain true under this new definition of quasi-hyperbolicity.

In [2], Baeza proved the norm theorem for nonsingular quadratic forms. Recently in [9], the first author extended this theorem to the case of totally singular quadratic forms by using the (first) notion of quasi-hyperbolicity for such quadratic forms (the new notion above also works since the situation was reduced to inseparable quadratic extensions where in this case the two notions of quasi-hyperbolicity coincide; cf. Proposition 2.5).

The aim of this paper is to discuss the norm theorem in the case of semisingular quadratic forms. Our first result is the following theorem which gives necessary conditions in order that an irreducible polynomial is a norm of a semisingular quadratic form, and proves that these conditions are sufficient if the polynomial is given by a quadratic form which represents 1:

Theorem 1.1. *Let φ be an anisotropic semisingular quadratic form over F, and let $p \in F[x_1, \dots, x_n]$ be an irreducible polynomial.*

(1) *If p is a norm of φ over $F(x_1, \dots, x_n)$, then the following conditions hold*:

(i) *p is inseparable which means that $\partial p/\partial x_i = 0$ for $1 \leqslant i \leqslant n$. In particular, if p is given by a quadratic form ψ, then ψ is totally singular.*

(ii) $i_W(\varphi_{F(p)}) = \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2}$ *and* $\mathrm{ql}(\varphi)$ *is quasi-hyperbolic over* $F(p)$.

(2) *Conversely, and if p is given by a quadratic form ψ which represents 1, then the following statements are equivalent*:

(i) *p is a norm of φ over $F(x_1, \dots, x_n)$.*

(ii) *ψ is totally singular,* $i_W(\varphi_{F(p)}) = \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2}$ *and* $\mathrm{ql}(\varphi)$ *is quasi-hyperbolic over* $F(p)$.

(iii) *ψ is totally singular and* $i_t(\varphi_{F(p)}) = \frac{\dim\varphi}{2}$.

Let us note that statement (2) of this theorem is still open for normed polynomials which are not necessary given by quadratic forms. Moreover, the condition $i_W(\varphi_{F(p)}) = \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2}$ given in statement (1)(ii) does not mean, in general, that a quadratic form R satisfying $\varphi \simeq R \perp \mathrm{ql}(\varphi)$ is hyperbolic over $F(p)$. Here is an example:

Example 1.2. Let a, b, c be variables over a field F_0 of characteristic 2, and let

$$\varphi = \left[1 + c^{-1}, a^{-1}\right] \perp b\left[1, a^{-1}\right] \perp c^{-1}\left(\langle 1\rangle \perp \langle b\rangle\right).$$

Then φ is anisotropic over $F = F_0(a, b, c)$, and admits $p = x^2 + b$ as a norm over $F(x)$, but $[1 + c^{-1}, a^{-1}] \perp b[1, a^{-1}]$ is not hyperbolic over $F(p)$.

Proof. Since $[1 + c^{-1}, a^{-1}] \perp \langle c^{-1}\rangle \simeq [1, a^{-1}] \perp \langle c^{-1}\rangle$, we get that $\varphi \simeq [1, a^{-1}] \perp b[1, a^{-1}] \perp c^{-1}(\langle 1\rangle \perp \langle b\rangle)$, and thus p is a norm of φ over $F(x)$. Moreover, φ is anisotropic over F since $[1, a^{-1}] \perp b[1, a^{-1}]$ and $\langle 1\rangle \perp \langle b\rangle$ are also anisotropic over $F_0(a, b)$ [9, Lemma 3.1]. The polynomial p is not a norm of $[1 + c^{-1}, a^{-1}] \perp b[1, a^{-1}]$, otherwise this form would be hyperbolic over $F(\sqrt{b})$, and then $[c^{-1}, a^{-1}]$ would be also hyperbolic over $F(\sqrt{b})$, which is not possible. □

Another important theorem in the theory of function fields of quadratic forms is the Cassels–Pfister subform theorem. In characteristic different from 2, this theorem is a consequence of a representation result by Pfister [10, Theorem 3.7, p. 151], and the norm theorem by Knebusch [4]. It asserts that if φ and ψ are anisotropic quadratic forms such that φ becomes hyperbolic over $F(\psi)$, then ψ is similar to a subform of φ. Of course, the converse of this statement fails in general. Recently, the first author and Hoffmann extended this theorem to the case of nonsingular quadratic forms φ [3, Theorem 4.2(i)]. However in this case, we use the domination relation (see Definition 2.1) instead of the subform relation since we don't exclude the case of singular quadratic forms ψ. Still in characteristic 2, the first author proved the subform theorem for totally singular quadratic forms φ by using the notion of quasi-hyperbolicity [6, Theorem 1.3].

Here we extend the subform theorem to the case of semisingular quadratic forms as follows:

Proposition 1.3. *Let $\varphi = R \perp \mathrm{ql}(\varphi)$ and ψ be anisotropic quadratic forms such that φ is semisingular and becomes quasi-hyperbolic over $F(\psi)$. Then the following conditions hold*:
(1) *ψ is totally singular.*
(2) *For any scalars $\alpha \in D_F(\psi)$, $\beta \in D_F(R)$ and $\gamma \in D_F(\mathrm{ql}(\varphi))$, there exists a nonsingular form R' such that $\varphi \simeq R' \perp \mathrm{ql}(\varphi)$ and ψ is dominated* (*cf. Definition* 2.1) *by the quadratic forms $\alpha\beta R'$ and $\alpha\gamma\mathrm{ql}(\varphi)$. In particular,* $\dim\psi \leqslant \min\{\frac{\dim R}{2}, \dim \mathrm{ql}(\varphi)\}$.

From this proposition, we know that for an anisotropic semisingular quadratic form φ, the quasi-hyperbolicity of $\varphi_{F(\psi)}$ implies in particular that ψ is dominated by $\mathrm{ql}(\varphi)$, up to a scalar, and thus it is totally singular. The following proposition deals with the case where ψ is similar to $\mathrm{ql}(\varphi)$:

Proposition 1.4. *Let φ be an anisotropic semisingular form. Then the following statements are equivalent*:
(1) *φ is quasi-hyperbolic over $F(\mathrm{ql}(\varphi))$.*
(2) *$\mathrm{ql}(\varphi)$ is similar to a quasi-Pfister form* (*i.e.,* $\mathrm{ql}(\varphi)$ *is isometric to the quadratic form $v \mapsto B(v, v)$, where B is similar to a bilinear Pfister form*)[1], *and*

$$\varphi \sim \left(\perp_{i=1}^{n} B \otimes [a_i, b_i] \right) \perp \mathrm{ql}(\varphi)$$

for some scalars $a_i, b_i \in F$, $1 \leqslant i \leqslant n$, where $\sim$ denotes the Witt-equivalence (*see below*), *and $\otimes$ is the action of the Witt ring of bilinear forms on the Witt group of nonsingular quadratic forms.*

2. PRELIMINARIES

For a quadratic form φ, we denote by $D_F(\varphi)$ the set of scalars in F^* represented by φ. Two quadratic forms φ and φ' are Witt-equivalent, denoted by $\varphi \sim \varphi'$, if $\varphi \perp m \times \mathbb{H} \simeq \varphi' \perp n \times \mathbb{H}$ for some integers m, n. For K/F a field extension and φ a quadratic form over F, we denote by φ_K the quadratic form $\varphi \otimes K$. To simplify notation, we denote by $\langle c_1, \ldots, c_s \rangle$ the totally singular form $\langle c_1 \rangle \perp \cdots \perp \langle c_s \rangle$.

For $a, b, c, d \in F$, the following isometries are well-known and easy to check:

$$(\star) \qquad \begin{cases} [a, b] \perp [c, d] \simeq [a + c, b] \perp [c, b + d], \\ c[a, b] \simeq [ca, c^{-1}b] \quad \text{if } c \neq 0, \\ [a, b] \perp \langle c \rangle \simeq [a + c, b] \perp \langle c \rangle. \end{cases}$$

[1] We refer to [7] and [3, Section 8] for details on quasi-Pfister forms.

We recall the domination relation between quadratic forms:

Definition 2.1 [3]. Let φ and φ' be quadratic forms with underlying vector spaces V and W, respectively. We say that φ is dominated by φ', denoted by $\varphi \prec \varphi'$, if there exists an injective F-linear map $t : (\varphi, V) \to (\varphi', W)$ such that $\varphi'(t(v)) = \varphi(v)$ for any $v \in V$.

The following proposition gives an equivalent definition of the domination relation:

Proposition 2.2 [3, Lemma 3.1]. *Let φ and φ' be two quadratic forms. Then, the following statements are equivalent*:

(1) $\varphi \prec \varphi'$.
(2) *There exist nonsingular forms Q, R, nonnegative integers $s' \leqslant s \leqslant t$, and scalars $c_1, \ldots, c_t, d_1, \ldots, d_{s'} \in F$ such that*:
 (i) $\varphi \simeq R \perp \langle c_1, \ldots, c_s \rangle$.
 (ii) $\varphi' \simeq Q \perp R \perp [c_1, d_1] \perp \cdots \perp [c_{s'}, d_{s'}] \perp \langle c_{s'+1}, \ldots, c_t \rangle$.

For the proof of Theorem 1.1, we need some preparatory results. First, for statement (1) we will be based on the following two lemmas:

Lemma 2.3. *Let φ be a quadratic form not totally singular and $p \in F[x_1, \ldots, x_n]$ be an irreducible polynomial which is a norm of φ. Then $i_W(\varphi_{F(p)}) \geqslant 1$ and $\mathrm{ql}(\varphi)_{F(p)}$ is quasi-hyperbolic.*

Proof. Set $\varphi = [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \mathrm{ql}(\varphi)$. Let $A = F[x_1, \ldots, x_n]$ and $K = F(x_1, \ldots, x_n)$. Since $R \perp \mathrm{ql}(\varphi) \simeq pR \perp p\mathrm{ql}(\varphi)$, it follows from the uniqueness of the quasilinear part that $\mathrm{ql}(\varphi) \simeq p\mathrm{ql}(\varphi)$. By [9, Theorem 1.1], $\mathrm{ql}(\varphi)$ is quasi-hyperbolic over $F(p)$. Moreover, by [7, Lemma 2.1], there exists a subform S of $\mathrm{ql}(\varphi)$ such that $(\mathrm{ql}(\varphi)_{F(p)})_{\mathrm{an}} \simeq S_{F(p)}$. $\square$

Claim. $\mathrm{ql}(\varphi)_K \simeq S_K \perp pS_K$.

Set $S = \langle c_1, \ldots, c_s \rangle$. Since $c_1, \ldots, c_s, pc_1, \ldots, pc_s \in D_K(\mathrm{ql}(\varphi)_K)$, it suffices to prove that $c_1, \ldots, c_s, pc_1, \ldots, pc_s$ are K^2-linearly independent [7, Lemma 2.1]. In fact, let $q_1, \ldots, q_s, q'_1, \ldots, q'_s \in K$, not all zero, such that

$$(2) \qquad \sum_{i=1}^{s} c_i q_i^2 + p \sum_{i=1}^{s} c_i q_i'^2 = 0.$$

We may suppose that $q_1, \ldots, q_s, q'_1, \ldots, q'_s \in A$ and p does not divide all of them. We extend (2) to $F(p)$ to get $\sum_{i=1}^{s} c_i \overline{q_i}^2 = 0 \in F(p)$. Since $S_{F(p)}$ is anisotropic, it follows that $q_i = r_i p$ for some $r_i \in A$ $(1 \leqslant i \leqslant s)$. We substitute in (2), we simplify by p, and we extend to $F(p)$ to get $\sum_{i=1}^{s} c_i \overline{q_i'}^2 = 0 \in F(p)$. Again, the anisotropy of $S_{F(p)}$ implies that p divides $q_1', \ldots, q_s'$, a contradiction to the choice of $q_1, \ldots, q_s, q_1', \ldots, q_s'$. Hence the claim.

Since $pa_1 \in D_K(\varphi_K)$, we may write

$$(3) \qquad \sum_{i=1}^{r} \left(a_i u_i^2 + u_i u_i' + b_i u_i'^2\right) + \sum_{i=1}^{s} c_i v_i^2 + p \sum_{i=1}^{s} c_i w_i^2 = (pa_1) a^2$$

for some $u_i, u_i' \in A$ $(1 \leqslant i \leqslant r)$, $v_j, w_j \in A$ $(1 \leqslant j \leqslant s)$ and $a \in A$. We may suppose that p does not divide all the polynomials u_i, u_i', v_j, w_j, a $(1 \leqslant i \leqslant r;\ 1 \leqslant j \leqslant s)$.

- Suppose that there exists $i \in \{1, \ldots, r\}$ such that $(u_i, u_i') \notin (pA)^2$. Then, the vector $(\bar{u}_1, \bar{u}_1', \ldots, \bar{u}_r, \bar{u}_r', \bar{v}_1, \ldots, \bar{v}_s, \overline{w}_1, \ldots, \overline{w}_s) \in F(p)^{2r+2s}$ does not belong to the radical of $\varphi_{F(p)}$, and thus by (3) $i_W(\varphi_{F(p)}) \geqslant 1$.
- Suppose that $(u_i, u_i') \in (pA)^2$ for each $i \in \{1, \ldots, r\}$. We extend (3) to $F(p)$ to get $\sum_{i=1}^{s} c_i \overline{v_i}^2 = 0 \in F(p)$. Since $S_{F(p)}$ is anisotropic, we conclude that p divides $v_1, \ldots, v_s$. Now, since $(u_i, u_i') \in (pA)^2$ for any $i \in \{1, \ldots, r\}$ and p divides $v_1, \ldots, v_s$, we deduce again from (3) that

$$a_1 \bar{a}^2 + \sum_{i=1}^{s} c_i \overline{w_i}^2 = 0 \in F(p).$$

Hence, $S \perp \langle a_1 \rangle$ is isotropic over $F(p)$ since p does not divide at least one of $w_1, \ldots, w_s, a$. Since $S_{F(p)}$ is anisotropic, it follows that $([a_1, b_1] \perp S)_{F(p)} \simeq \mathbb{H} \perp S_{F(p)}$. In particular, $i_W(\varphi_{F(p)}) \geqslant 1$. $\square$

Lemma 2.4. *Let $p \in F[x_1, \ldots, x_n]$ be an irreducible polynomial, and let $\varphi = R \perp \mathrm{ql}(\varphi)$ be an anisotropic quadratic form such that $\dim R > 0$ and $R_{F(p)}$ is not hyperbolic. Then p stays irreducible over $F(\varphi)$.*

Proof. Set $R = a_1[1, b_1] \perp \cdots \perp a_r[1, b_r]$. Since $R_{F(p)}$ is not hyperbolic, there exists $i \in \{1, \ldots, r\}$ such that $[1, b_i]_{F(p)}$ is anisotropic. Let $\varphi' = [1, b_i]$ and suppose that p is reducible over $F(\varphi)$. Since $F(\varphi')(\varphi)/F(\varphi')$ is purely transcendental, p is reducible over $F(\varphi')$, and thus it is also reducible over the separable quadratic extension $F(\beta)$ given by $\beta^2 + \beta + b_i = 0$.

Let $q, r \in F(\beta)[x_1, \ldots, x_n]$ such that $0 < \deg q < \deg p$ and $p = qr$ (deg means the total degree). We can write $q = q_1 + \beta q_2$ and $r = r_1 + \beta r_2$ with $q_1, q_2, r_1, r_2 \in F[x_1, \ldots, x_n]$. The relation $p = qr$ implies the following:

$$\begin{cases} p = q_1 r_1 + b_i q_2 r_2 & (1), \\ q_2 r_2 = q_1 r_2 + q_2 r_1 & (2). \end{cases}$$

We multiply (1) and (2) by q_2 and q_1, respectively, and we add the new relations to get $pq_2 = r_2(q_1^2 + q_1 q_2 + b_i q_2^2)$. Hence, by reason of degree p divides $q_1^2 + q_1 q_2 + b_i q_2^2$. In particular, $\bar{q}_1^2 + \bar{q}_1 \bar{q}_2 + b_i \bar{q}_2^2 = 0 \in F(p)$. Since p does not divide q_1 and q_2, we deduce that $[1, b_i]$ is isotropic over $F(p)$, a contradiction. $\square$

For the proof of statement (2) of Theorem 1.1 we need other results. The following proposition gives some equivalent descriptions of quasi-hyperbolicity over function fields of quadratic forms:

Proposition 2.5. *Let φ and ψ be anisotropic quadratic forms. Then:*
(1) $i_t(\varphi_{F(\psi)}) \leqslant [\frac{\dim\varphi}{2}]$, *where* $[n]$ *denotes the integer part of* n.
(2) *Suppose* $\dim \mathrm{ql}(\varphi) > 0$ *and even. Then the following statements are equivalent*:

(i) φ *is quasi-hyperbolic over* $F(\psi)$,
(ii) $i_t(\varphi_{F(\psi)}) = \frac{\dim\varphi}{2}$,
(iii) $i_W(\varphi_{F(\psi)}) = \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2}$ *and* $i_d(\varphi_{F(\psi)}) = \frac{\dim \mathrm{ql}(\varphi)}{2}$.

Proof. (1) We have

$$\begin{aligned} i_t(\varphi_{F(\psi)}) &= i_W(\varphi_{F(\psi)}) + i_d(\varphi_{F(\psi)}) \\ &\overset{(\mathbf{a})}{=} i_W(\varphi_{F(\psi)}) + i_d(\mathrm{ql}(\varphi)_{F(\psi)}) \\ &\overset{(\mathbf{b})}{\leqslant} \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2} + \left[\frac{\dim \mathrm{ql}(\varphi)}{2}\right] = \left[\frac{\dim\varphi}{2}\right], \end{aligned}$$

where (**a**) follows from the uniqueness of the quasilinear part, and (**b**) follows from the inequality $i_W(\varphi_{F(\psi)}) \leqslant \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2}$ and [6, Corollary 2.13].

(2) (i) $\Rightarrow$ (ii) Follows from (1) and the definition of the quasi-hyperbolicity.

(ii) $\Rightarrow$ (iii) Follows from the inequalities $i_d(\varphi_{F(\psi)}) \leqslant \frac{\dim \mathrm{ql}(\varphi)}{2}$ and $i_W(\varphi_{F(\psi)}) \leqslant \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2}$.

(iii) $\Rightarrow$ (i) This is obvious. □

The following lemma can be proved as in [3, the end of the proof of Lemma 5.4]. To keep our paper self-contained we reproduce a proof:

Lemma 2.6. *Let φ be an anisotropic quadratic form over F. Suppose that φ becomes isotropic over $F(\sqrt{d})$ with $d \in F^* - F^{*2}$. Then there exists a scalar $\alpha \in F^*$ such that $\alpha\langle 1, d\rangle \prec \varphi$.*

Proof. Let V be the underlying vector space of φ. An easy computation shows that the isotropy of $\varphi_{F(\sqrt{d})}$ implies the existence of two vectors $u, v \in V$, not both zero, such that $\varphi(u) = d\varphi(v)$ and $B_\varphi(u, v) = 0$, where B_φ is the bilinear form associated to φ. Since φ is anisotropic, we have $\varphi(u), \varphi(v) \neq 0$. The vectors u, v are linearly independent, otherwise there would be a scalar $x \in F^*$ such that $u = xv$, and thus $\varphi(v)(x^2 + d) = 0$ which contradicts $d \notin F^{*2}$. From the definition of the domination relation it is clear that $\alpha\langle 1, d\rangle \prec \varphi$ where $\alpha = \varphi(v)$. □

A tricky result for the proof of statement (2) is the following proposition:

Proposition 2.7. *Let φ be a quadratic form which is not totally singular such that $i_W(\varphi_{F(\sqrt{d})}) = \frac{\dim\varphi - \dim \mathrm{ql}(\varphi)}{2}$ with $d \in F^* - F^{*2}$. Then there exists a nonsingular quadratic form R' over F which admits $x^2 + d$ as a norm, and satisfies $\varphi \sim R' \perp \mathrm{ql}(\varphi)$.*

Proof. Let R be a nonsingular form such that $\varphi = R \perp \mathrm{ql}(\varphi)$. Let $L = F(\sqrt{d})$ and $r = \frac{\dim R}{2}$.

1. *Reduction to the case where* $\mathrm{ql}(\varphi)_L$ *is anisotropic.* If $\dim \mathrm{ql}(\varphi) = 0$, then the proposition follows from the norm theorem by Baeza [2]. Suppose that $\dim \mathrm{ql}(\varphi) > 0$. By [7, Lemma 2.1], there exists a subform S of $\mathrm{ql}(\varphi)$ such that $(\mathrm{ql}(\varphi)_L)_{\mathrm{an}} \simeq S_L$. Hence, by our hypothesis

$$(R \perp S)_L \perp j \times \langle 0\rangle \simeq r \times \mathbb{H} \perp S_L \perp j \times \langle 0\rangle,$$

where $j = i_d(\mathrm{ql}(\varphi)_L)$. It follows from [3, Lemma 2.6] that

$$(R \perp S)_L \simeq r \times \mathbb{H} \perp S_L,$$

and thus we are reduced to the case where $\mathrm{ql}(\varphi)_L$ is anisotropic.

2. *Case where* $\mathrm{ql}(\varphi)_L$ *is anisotropic.* By Lemma 2.6, there exists $\alpha \in F^*$ such that $\alpha\langle 1, d\rangle \prec \varphi$. Since $\mathrm{ql}(\varphi)_L$ is anisotropic, it follows from Proposition 2.2 that $\alpha\langle 1, d\rangle \simeq \langle u, v\rangle$ for suitable scalars $u, v \in F^*$, and one of the following cases holds:

Case 1. $\varphi \simeq [u, x_0] \perp [v, y_0] \perp R_0 \perp \mathrm{ql}(\varphi)$ for $x_0, y_0 \in F$ and a nonsingular form R_0.

Case 2. $\varphi \simeq [u, x_1] \perp R_1 \perp \langle v\rangle \perp S_1$ for $x_1 \in F$, a nonsingular form R_1 and a totally singular form S_1.

We proceed by induction on r. Since $\langle u, v\rangle$ is isotropic over L, we get $uv \in L^2$. Hence, $[u, x_i] \perp [v, v^{-1}ux_i]$ is isotropic over L $(i = 0, 1)$, and thus it is hyperbolic since its Arf invariant is trivial. In particular, $x^2 + d$ is a norm of $[u, x_i] \perp [v, v^{-1}ux_i]$.

(1) Suppose $r = 1$. Then we are in Case 2 with $R_1 = 0$. Since $[v, v^{-1}ux_1] \perp \langle v\rangle \sim \langle v\rangle$, we get $\varphi \sim [u, x_1] \perp [v, v^{-1}ux_1] \perp \langle v\rangle \perp S_1$. Hence the claim.

(2) Suppose $r \geqslant 2$:

(i) Suppose we are in Case 1. Since $([u, x_0] \perp [v, v^{-1}ux_0])_L \sim 0$, it follows from the first isometry in $(\star)$ that $([u, x_0] \perp [v, y_0])_L \simeq \mathbb{H} \perp [v, y_0 + v^{-1}ux_0]_L$. Hence,

$$\varphi_L \simeq \mathbb{H} \perp \big([v, y_0 + v^{-1}ux_0] \perp R_0 \perp \mathrm{ql}(\varphi)\big)_L.$$

By Witt cancellation [4, Proposition 1.2],

$$i_W\big(\big([v, y_0 + v^{-1}ux_0] \perp R_0 \perp \mathrm{ql}(\varphi)\big)_L\big) = r - 1 = \frac{\dim R_0 + 2}{2} < r.$$

By induction, there exists a nonsingular form R'_0 over F which admits $x^2 + d$ as a norm, and satisfies

$$[v, y_0 + v^{-1}ux_0] \perp R_0 \perp \mathrm{ql}(\varphi) \sim R'_0 \perp \mathrm{ql}(\varphi).$$

This equivalence implies the following

$$[v, y_0] \perp R_0 \perp \mathrm{ql}(\varphi) \sim [v, v^{-1}ux_0] \perp R'_0 \perp \mathrm{ql}(\varphi),$$

and thus $\varphi \sim [u, x_0] \perp [v, v^{-1}ux_0] \perp R'_0 \perp \mathrm{ql}(\varphi)$. Hence the claim since $x^2 + d$ is a norm of $[u, x_0] \perp [v, v^{-1}ux_0]$.

(ii) Suppose we are in Case 2. Since $uv \in L^2$, we get $\varphi_L \simeq \mathbb{H} \perp (R_1 \perp \langle v\rangle \perp S_1)_L$. By Witt cancellation,

$$i_W\big((R_1 \perp \langle v\rangle \perp S_1)_L\big) = r - 1 = \frac{\dim R_1}{2} < r.$$

By induction, there exists a nonsingular form R'_1 over F, which admits $x^2 + d$ as a norm and satisfies

$$R_1 \perp \langle v\rangle \perp S_1 \sim R'_1 \perp \langle v\rangle \perp S_1.$$

Since $[v, v^{-1}ux_1] \perp \langle v\rangle \sim \langle v\rangle$, it follows that

$$\varphi \sim [u, x_1] \perp [v, v^{-1}ux_1] \perp R'_1 \perp \langle v\rangle \perp S_1.$$

Hence the claim since $x^2 + d$ is a norm of $[u, x_1] \perp [v, v^{-1}ux_1]$. $\square$

3. PROOF OF THEOREM 1.1

Let R be a nonsingular form such that $\varphi \simeq R \perp \mathrm{ql}(\varphi)$, and let $p \in F[x_1, \dots, x_n]$ be an irreducible polynomial which is a norm of φ.

(1) We have to prove that p is inseparable, $i_W(\varphi_{F(p)}) = \frac{\dim R}{2}$ and $\mathrm{ql}(\varphi)_{F(p)}$ is quasi-hyperbolic.

Indeed, by Lemma 2.3 $\mathrm{ql}(\varphi)_{F(p)}$ is quasi-hyperbolic, and thus p is inseparable [9, Corollary 2.9]. It remains to prove that $i_W(\varphi_{F(p)}) = \frac{\dim R}{2}$. We proceed by induction on $\dim R$.

If $\dim R = 2$, then the claim follows from Lemma 2.3. Suppose $\dim R > 2$, and let $K = F(\varphi)$. The form $\mathrm{ql}(\varphi)_K$ is anisotropic [5, Corollaire 3.3]. Hence, $\varphi_K \simeq i \times [0, 0] \perp R' \perp \mathrm{ql}(\varphi)_K$ for some positive integer i and a nonsingular form R' over K (possibly the zero form). If $R_{F(p)}$ is hyperbolic, then we are done. So let us assume that $R_{F(p)}$ is not hyperbolic. By Lemma 2.4, p_K is irreducible. Obviously, over the field $K(x_1, \dots, x_n)$ we have the isometry

$$i \times [0, 0] \perp R' \perp \mathrm{ql}(\varphi) \simeq i \times [0, 0] \perp pR' \perp p\mathrm{ql}(\varphi).$$

By Witt cancellation, we have $R' \perp \mathrm{ql}(\varphi) \simeq pR' \perp p\mathrm{ql}(\varphi)$ (still over $K(x_1, \dots, x_n)$). Since $\dim R' < \dim R$, we get by induction that $i_W((R' \perp \mathrm{ql}(\varphi))_{K(p)}) = \frac{\dim R'}{2}$.

Hence, $i_W(\varphi_{K(p)}) = i + \frac{\dim R'}{2} = \frac{\dim R}{2}$. By Lemma 2.3, $i_W(\varphi_{F(p)}) \geqslant 1$ and thus $F(p)(\varphi)/F(p)$ is purely transcendental. Since $K(p) = F(p)(\varphi)$, we conclude that $i_W(\varphi_{F(p)}) = \frac{\dim R}{2}$.

(2) Suppose that p is given by a quadratic form ψ which represents 1. Set $\psi = \langle 1, a_2, \ldots, a_n \rangle$ and $F(\psi) = L(\sqrt{d})$, where $L = F(x_2, \ldots, x_n)$ and $d = \sum_{i=2}^{n} a_i x_i^2$. After statement (1) and Proposition 2.5, it remains to prove (ii) $\Rightarrow$ (i).

Suppose that $i_W(\varphi_{F(p)}) = \frac{\dim R}{2}$ and $\mathrm{ql}(\varphi)_{F(p)}$ is quasi-hyperbolic. By Proposition 2.7, there exists a nonsingular form R' over L such that $\varphi_L \sim R' \perp \mathrm{ql}(\varphi)_L$ and $x_1^2 + d$ is a norm of R'. Moreover, since $\mathrm{ql}(\varphi)_{F(p)}$ is quasi-hyperbolic, we deduce from [9, Theorem 1.1] that $x_1^2 + d$ is also a norm of $\mathrm{ql}(\varphi)_{L(x_1)}$. Hence, $x_1^2 + d = \psi(x_1, \ldots, x_n) = p$ is a norm of φ over $F(x_1, \ldots, x_n)$. $\square$

4. PROOF OF PROPOSITION 1.3

Let $\varphi = R \perp \mathrm{ql}(\varphi)$ be an anisotropic semisingular form, and let ψ be an anisotropic form such that $\varphi_{F(\psi)}$ is quasi-hyperbolic. Let $\alpha \in D_F(\psi)$, $\beta \in D_F(R)$ and $\gamma \in D_F(\mathrm{ql}(\varphi))$. We will prove the existence of a nonsingular form R' such that $\varphi \simeq R' \perp \mathrm{ql}(\varphi)$ and ψ is dominated by $\alpha\beta R'$ and $\alpha\gamma \mathrm{ql}(\varphi)$.

Without loss of generality, we may suppose $\alpha = \beta = 1$. Since the condition $i_t(\varphi_{F(\psi)}) \geqslant \frac{\dim \varphi}{2}$ implies the quasi-hyperbolicity of $\mathrm{ql}(\varphi)_{F(\psi)}$ (Proposition 2.5), the form ψ is totally singular and dominated by $\gamma \mathrm{ql}(\varphi)$ [6, Theorem 1.3].

Set $R = [1, b_1] \perp Q$ for some nonsingular form Q, and $\psi = \langle a_1, \ldots, a_n \rangle$ $(a_1 = 1)$. To complete the proof we will check that

$$\varphi \simeq \left(\perp_{i=1}^{n} [a_i, b_i] \right) \perp Q' \perp \mathrm{ql}(\varphi)$$

for some scalars $a_i, b_i \in F^*$ $(1 \leqslant i \leqslant n)$, and a nonsingular form Q', and then $(\perp_{i=1}^{n} [a_i, b_i]) \perp Q'$ will be the form R' that we need.

There is nothing to prove for $n = 1$. Suppose that $n > 1$, and that we have proved the isometry

$$\varphi \simeq \left(\perp_{i=1}^{r} [a_i, b_i'] \right) \perp Q'' \perp \mathrm{ql}(\varphi)$$

for some $r < n$ and a nonsingular form Q''.

By Theorem 1.1(2), the polynomial $p = \sum_{i=1}^{n} a_i x_i^2$ is a norm of φ over $F(x_1, \ldots, x_n)$. We take the F-place λ from $F(x_1, \ldots, x_n)$ to $F(x_1, \ldots, x_{r+1})$ defined by:

$$\lambda(x_i) = \begin{cases} x_i & \text{if } 1 \leqslant i \leqslant r+1, \\ 0 & \text{otherwise.} \end{cases}$$

If A denotes the ring of λ, and m its maximal ideal, then A/m can be viewed as a subfield of $F(x_1, \ldots, x_{r+1})$. In particular, φ remains anisotropic over A/m, and then it has nearly good reduction in the sense of [4, beginning of page 289]. By [4, Proposition 2.9], and after taking the specialization of φ and $p\varphi$ under λ, we deduce that $q = \sum_{i=1}^{r+1} a_i x_i^2$ is a norm of φ over $F(x_1, \ldots, x_{r+1})$. Since $1 \in D_F(\varphi)$, the form φ represents q over $F(x_1, \ldots, x_{r+1})$. By [1, Lemma 3.3], the form

$$\big(\perp_{i=1}^{r} \langle a_i \rangle\big) \perp Q'' \perp \mathrm{ql}(\varphi)$$

represents $a_{r+1}x_{r+1}^2$ over $F(x_{r+1})$. In particular, the form $(\perp_{i=1}^{r+1} \langle a_i \rangle) \perp Q'' \perp \mathrm{ql}(\varphi)$ is isotropic over F. Since $(\perp_{i=1}^{r} \langle a_i \rangle) \perp Q'' \perp \mathrm{ql}(\varphi)$ is anisotropic over F (because it is dominated by φ), there exist $a \in D_F(Q'') \cup \{0\}$, $b \in D_F(\mathrm{ql}(\varphi)) \cup \{0\}$ and $e_1, \ldots, e_r \in F$ such that $a_{r+1} = a + b + \sum_{i=1}^{r} a_i e_i^2$.

We claim that $a \neq 0$. Indeed, if $a = 0$ then $b \neq 0$ since ψ is anisotropic. Hence, $D_F(\psi) \cap D_F(\mathrm{ql}(\varphi)) \neq \emptyset$. Since $\mathrm{ql}(\varphi)_{F(\psi)}$ is quasi-hyperbolic, it follows from [9, Theorem 1.3] that $\psi \prec \mathrm{ql}(\varphi)$. In particular, $1 \in D_F(\mathrm{ql}(\varphi))$, and thus φ is isotropic since $1 \in D_F(R)$, a contradiction.

Since $a \in D_F(Q'')$, we may write $Q'' \simeq [a, c] \perp Q'$ for some nonsingular form Q' and $c \in F^*$. Hence

$$\varphi \simeq \big(\perp_{i=1}^{r} [a_i, b_i']\big) \perp [a, c] \perp Q' \perp \mathrm{ql}(\varphi).$$

By using the relation $a_{r+1} = a + b + \sum_{i=1}^{r} a_i e_i^2$, and the isometries described in $(\star)$, it is easy to see that

$$\varphi \simeq \big(\perp_{i=1}^{r+1} [a_i, b_i]\big) \perp Q' \perp \mathrm{ql}(\varphi)$$

where

$$b_i = \begin{cases} b_i' + c e_i^2 & \text{if } i \in \{1, \ldots, r\}, \\ c & \text{if } i = r + 1. \end{cases}$$

If $n = r + 1$ then we are done. If not, we reproduce successively the same proof for the scalars $a_{r+2}, \ldots, a_n$ to get the desired conclusion.

5. PROOF OF PROPOSITION 1.4

Let $\varphi = R \perp \mathrm{ql}(\varphi)$ be an anisotropic semisingular form, and $K = F(\mathrm{ql}(\varphi))$.

(1) $\Rightarrow$ (2) Suppose that φ becomes quasi-hyperbolic over K. By Theorem 1.1, the form $\mathrm{ql}(\varphi)$ is quasi-hyperbolic over K. It follows from [6, Corollary 1.8] that $\mathrm{ql}(\varphi)$ is similar to a quasi-Pfister form. Let $B = \langle a_1 \rangle_b \perp \cdots \perp \langle a_s \rangle_b$ be similar to a bilinear Pfister form such that $\mathrm{ql}(\varphi)$ is isometric to the quadratic form $v \mapsto B(v, v)$ (here $\langle a \rangle_b$ denotes the bilinear form axy for $a \in F^*$). Hence, $\mathrm{ql}(\varphi) \simeq \langle a_1, \ldots, a_s \rangle$. By the same proof as for Proposition 1.3, we get

$$\varphi \simeq \alpha\big(\perp_{i=1}^{s} a_i[1, b_i]\big) \perp Q \perp \mathrm{ql}(\varphi)$$

for scalars $\alpha, b_1, \ldots, b_s \in F^*$ and a nonsingular form Q. In particular,

$$\varphi \sim \alpha\big(\perp_{i=1}^{s} a_i[1, b_1]\big) \perp \alpha\big(\perp_{i=2}^{s} a_i[1, b_1 + b_i]\big) \perp Q \perp \mathrm{ql}(\varphi).$$

Let $\varphi' = (\alpha(\perp_{i=2}^{s} a_i[1, b_1 + b_i]) \perp Q \perp \mathrm{ql}(\varphi))_{\mathrm{an}}$. The form $\alpha(\perp_{i=1}^{s} a_i[1, b_1])$ is just $\alpha B \otimes [1, b_1]$, which is similar to a Pfister form. Hence, $B \otimes [1, b_1]$ is hyperbolic over K as B is isotropic over K, and thus $\varphi_K \sim \varphi'_K$. Since φ_K is quasi-hyperbolic, the same thing holds for φ'_K. Since $\dim \varphi' - \dim \mathrm{ql}(\varphi') \leqslant \dim(\alpha(\perp_{i=2}^s a_i[1, b_1 + b_i]) \perp Q) < \dim R$, the claim follows by induction on $\dim R$.

(2) $\Rightarrow$ (1) Suppose that $\mathrm{ql}(\varphi)$ is similar to a quasi-Pfister form, and $\varphi \sim (\perp_{i=1}^n B \otimes [a_i, b_i]) \perp \mathrm{ql}(\varphi)$ for some $a_i, b_i \in F$ $(1 \leqslant i \leqslant n)$, where B is similar to a bilinear Pfister form and $\mathrm{ql}(\varphi)$ is isometric to the quadratic form given by $v \mapsto B(v, v)$.

Since B is isotropic over K, the form $B \otimes [a_i, b_i]$ is hyperbolic over K $(1 \leqslant i \leqslant n)$, and $\varphi_K \sim \mathrm{ql}(\varphi)_K$. Moreover, the form $\mathrm{ql}(\varphi)_K$ is quasi-hyperbolic [6, Corollary 1.8; 3]. Hence, the quasi-hyperbolicity of φ_K.

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